Some analysis on a first course in linear algebra

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Abstract: The aim of this paper is to analyze some topics about linear algebra course, along with researches and opinions which are original and useful. The considered topics are the content, textbooks, students' learning profiles, teaching methods, using computer programs, and connections with other mathematics courses .According to the main results of the analyses, it is an oversimplification to think that there is a unique right way to teach this course. Although many mathematicians could expect that the first linear algebra as if were the same everywhere, the reality is different from this idea. The recent editions of linear algebra textbooks are usually good materials for what is being taught at the introductory level. It seems that only expressing and showing of teacher may not significantly improve students' learning of an abstract course. In recent years, linear algebra researchers have formulated some efficient teaching methods in order to facilitate meaningful learning. Software provides helpful visualization in two or three dimensional vector spaces. By the creating interactive environment of the computer programs, students can explore with matrices, linear transformations and numerical representations. And finally, there is an obvious connection between linear algebra, calculus, differential equations, and statistics.
Keywords: learning and teaching linear algebra, textbook, computer program

Introduction

There is a general view expressed in the literature that students having problem with linear algebra course have very little understanding of the basic abstract concepts. Carlson (1993) stated that solving systems of linear equations and calculating products of matrices is easy for the students. However, when they get to subspaces, spanning, and linear independence students become confused and disoriented. Carlson (1993) further identified the reasons why certain topics in linear algebra are so difficult for students; presently linear algebra is taught far earlier and to less sophisticated students than before. The topics that create difficulties for students are concepts, not computational algorithms. Also, different algorithms are required to work with these ideas in different settings.

The effective ways to teach linear algebra is one of the main study subjects who focused on linear algebra teaching. When a teacher has taught a course a few years, it is very easy to slip into a routine teaching manner in this course. The textbook, the material, the lectures, the questions, all become familiar. The instructor has no hesitation what must be done each session of the course. In such a case, everybody knows the effective ways to teach linear algebra. But, it is suggested that there are deeper issues on the teaching the course that need a new modification (Carlson at al., 1993). Elementary linear algebra courses are taught large and very diverse students in the last years. How much emphasis should be done on theorems, proofs and applications? How abstract should the concepts be? How effect computer can be benefited? How connection is there with the other mathematics courses? And which teaching strategies are effective? These questions have revealed a wide interest in linear algebra teaching. There have been widespread discussions, workshops of experts, special sessions at the meetings and panels. Also, it has been presented many research articles and textbooks on linear algebra education. After all, what can be said about teaching linear algebra?

In this study, it is analyzed goals and content, students and teaching approaches, texts and computer software. By an educational perspective, nobody is professional the art of teaching linear algebra. It is not easy to say that there is a unique right teaching way for this course. Although many linear algebra teachers speak of the first linear algebra course as if it were the same everywhere, the reality is far, richer and more diverse (Harel, 1998). Even our student group described linear algebra courses for this study that reflect different goals, and approaches. The purpose of this article is to share some of these questions, along with resources and opinions.

The Content

The Linear Algebra Curriculum Study Group (LACSG) recommended a core curriculum for a first linear algebra course (Carlson at al., 1993). The group members first interviewed from a variety of linear algebra instructors, and they developed their report after much discussion about that needs linear algebra and what is reasonable to teach in a first course, and how computer software should affect the teaching. As a result of the recommendations proposed by LACSG, the emphasis in linear algebra was shifted to a matrix-oriented course concentrating on applications and reducing the emphasizing on abstraction of concepts. While this shift in focus is valuable to mathematics and non-mathematics majors, the relegation of abstraction to an "also ran" in comparison to applications is doing mathematics majors a great disservice. According to Alan Tucker (1993, p. 4), "linear algebra was positioned to be the first real mathematics course in the undergraduate mathematics curriculum because its theory is so well structured and comprehensive, yet requires limited mathematical prerequisites". It is the first class where undergraduates are expected to prove theorems and is thus a pivotal course with respect to their ability to conjecture and write coherent proofs. Tucker (1993, p.5) emphasizes that "A mastery of finite vector spaces, linear transformations, and their extensions to function spaces is essential for a practitioner or researcher in most areas of pure and applied mathematics". The content of many textbooks reflects the LACSG recommendations, and computer software (such as matlab and mathematica) has become more powerful (Howard, 1997; Richard, 1997; Kolman, 1999).

In 1998, a study group from the Park City Mathematics Institute (PCMI) considered the idea of trying to update this recommended curriculum, in light of how it is actually being used today (Day & Kalman, 1999). In the PCMI there were as many ways to construct a first linear algebra course as there were different departments' students for linear algebra, and they expect the first linear algebra course to play different roles in the their own curriculum. Some have a large population of engineering students, with an emphasis on physical science applications. It was soon clear for the study group that no single model curriculum would serve the needs of all these different approaches, and they abandoned any hope of either updating the LACSG recommendations.

What should be the content of the first linear algebra course? After the above discussions, we can say that a first step toward answering this question is to decide what curriculum model makes sense at your department. That means your department must pay attention to what topics you hope the students will learn.

Textbooks

Linear algebra did not really come to be recognized as a subject until the 1930's. Particularly influential in this process were the book of B. L. van der Waerden (1936) and the book of Garrett Birkhoff and Saunders MacLane (1941). Both were on "Modern Algebra" but included chapters on linear algebra. The separate linear algebra course became a standard part of the college mathematics curriculum in the United States in the 1950's and 1960's (Schneider & Barker, 1968). It appears that the Introductory linear algebra course was one of the first times it was offered there as a regular course in 1965 at Indiana University (Cowen, 1997).

From the 1970s until today, every concrete Linear Algebra textbook based approach starts with practical computations, such as Gaussian row reduction, or with applications such as systems of linear equations and progresses to the underlying concepts. It is driven by concrete forces and attempts to understand abstract concepts from examples (Anton, 1973). This approach has persisted into the 1990's (Lay, 1994).

I have reviewed eight recent editions of linear algebra books received from different publishers, to discern any general trends in the teaching of introductory linear algebra (Strang, 2005; Poole, 2007; Lawrence, Insel, & Friedberg, 2008). These texts are a good representative sample for what is being taught at the introductory level. Given that most authors of textbooks try to pack in enough material so as to make the text useable for more advanced linear algebra courses, it is not surprising that the general theory of vector spaces is introduced at some point towards the middle or the end of the text. Several authors introduce the language of the general theory early, first only in the context of R^n, as a transition from the concrete to the abstract.

We see in these books that traditional applications of linear algebra which have tended to come from Physics and Engineering are now augmented by simple applications from disciplines such as Economics, Biology, and Computer Science. It would be rare indeed to see a linear algebra text nowadays without links to calculator and computer technology and without some discussion of computational issues. Some texts envision the technology as a fully integrated component, and others as an add-on. The technology aspect appears either in the form of an accompanying Lab manual, a reference to web-based activities, or as "technology exercises" at the end of each chapter. However, all of the texts are written in a way that allows one to use them without any technological components.

Students' learning profiles

Who are linear algebra students? What are their goals? Why are they learning linear algebra? All of the subjects in linear algebra education, the issue of understanding how students learn is the one that made the greatest impact on the researchers (Dubinsky, 1997; Dorier, Robert, Robinet, & Rogalski, 2000).There is not a great deal of published literature on how students learn linear algebra. Guershon Harel has been studying some aspects of this for several years, and his papers provide some suggestions for linear algebra teachers (Katz, 1995; Harel, 1998). David Carlson (1993) presents an interesting hypothesis about the special difficulties that linear algebra presents for students, and Ed Dubinsky (1997) offers another point of view.

Many teachers accept that students have to construct their own knowledge in order to achieve meaningful learning. To be more accurate, we are, as teachers, not sure how to assist students in this construction, but it seems that simply showing what is true and telling students what we wish them to know is not generally sufficient. Several studies in calculus course have showed that students do not generally have a rich conceptual understanding of graphs and functions (Davis, & Vinner, 1996). It was analyzed that linearity and independence are also concepts with which most students struggle (Bogomonly, 1999). And it seems clear that simply doing a better job of telling and showing may not significantly improve their learning of such difficult topics.

The researchers (Day, & Kalman, 1999) from the Park City Mathematics Institute proposed that we should become better listeners. Specifically, they suggested that we select a few students and interview them in depth, instead of correcting their misconceptions. This means taking time to observe and listen carefully to what they are really doing when they think about linear algebra. Explore what the student is thinking, paying careful attention to the ideas behind what the student says. How does a student reach conclusions that we find wrong? Over time, such interviews may reveal some ways in the thinking and misconceptions of our students that lead us to a better understanding of how they learn.

As mathematicians, we are aware of the rich interconnections of different ideas and concepts in mathematics. We would like our students to learn how different ideas work together, supporting and validating each other. We know from experience that understanding of this kind is not acquired as a result of being told of each definition or principle. It develops through actively exploring a mathematical topic, discovering and rediscovering the interconnections until they become familiar and commonplace. But we who have developed understanding on this level risk forgetting the effort that came before: the missteps, false generalizations, incomplete and inconsistent conceptions. Meaningful learning is difficult to achieve and it rarely occurs unless students actively grapple with the ideas.

Teaching strategies

The usability of computers has forced mathematicians to rethink the way they are teaching mathematics. When a calculation can be operated quickly and satisfactorily by a computer program, one has to ask 'what is it that a student really needs to learn?' As a response, at the least, students need to develop critical thinking skills, to understand well the main concepts of mathematics and to be able to apply them in different situations.

In his study, Herrero defined five strategies to effective using of computer in the teaching linear algebra (Herrero, 2000). The following strategies are described: (1) exploration of new concepts through computer exercises; (2) teaching linear transformations as early as possible; (3) emphasis on geometry; (4) teaching to write mathematics through development of a portfolio; (5) using computer projects for motivation and applications. The purpose of each of these projects is to introduce students to a new subject in linear algebra through a hands-on approach. They are intended to provide motivation for new definitions, show the need for the new theorems, make conjectures, and realize the usefulness of the new theorems by applying them to solve various problems. Computer-based instructions may lower the quality of learning if too much emphasis is placed on individual work with the computer. Incorporating technology into mathematics teaching works best, when it is done with teaching strategies that benefit from critical thinking and increase communication between students and teachers.

What can linear algebra teachers do to enrich lecturing, in order to better facilitate meaningful learning? Guershon Harel and Larry Sowder (2003) used an intense lecture-discussion method in his linear algebra classes at Purdue, which could be described as a rich extension of lecture. He insists that the students participate in working through all concepts. He uses MATLAB in a brief but important way, to facilitate examples: together the class figures out what must be calculated, how to do it and what results are expected; then he does the calculation and they discuss whether the results were what they expected and how to verify them.

Day (1997) applied "Mazur's polling method" in linear algebra classes. The author, in particular, tried a few variations on the method. She used polling spontaneously and very quickly, to liven up lectures (Show hands. Who thinks that idea will work? Who doesn't?). She also used it more formally after many students missed the following question on an early test: "True or False?" *If A and B are invertible then A + B is invertible.* Before returning the graded tests, she asked the class to vote on this question, and about half said "true", half "false". Then she asked students to find someone who disagreed with them and discuss which answer was really correct. She moved around, listening and occasionally asking pointed questions, for about 5 minutes. Then they voted again and about %80 got the correct answer. Some of the students who understood correctly explained how they went about answering it. This exercise took about 15 minutes, but it was a productive way to get students to think about how to analyze such a question, and to see how effective it can be to look for really simple examples.

Using computer programs

Why and how to use technology is another question for linear algebra teaching. There are several different roles that technology can play in instruction, from eliminating computational drudgery in realistic applications, to providing environments for actively exploring the properties of mathematical structures and objects (Herrero, 2000). Linear algebra teachers have different views and experience using computer programs (MATLAB, Maple, Mathematica, Mathwright) in the lectures. Some of them assign computer projects to be done outside of class. Some use computer demos and examples to enrich lectures, and others rarely lecture at all, instead using software as a primary means for delivering mathematical material to the students, with a significant proportion of class time spent interacting with the computer.

The main purposes can be summarized from the different views about why and how to use software in teaching linear algebra are: for computation in meaningful applications; as a matrix calculator; as a direct focus of instruction; for visualization; to provide an environment for active exploration of mathematical structures; and to explore some of the limitations of floating point calculations.

It is possible to find many applications that will be of interest to students from just about any background. However, in most real world problems, the dimensions of the matrices make hand calculation completely inadequate. Even with relatively low dimensional problems, the overhead of hand calculation quickly becomes distracting or simply overwhelming. Some teachers use technology just to provide students first-hand experience with real applications in realistic settings.

Calculators and software like Matlab, Maple, and Mathematica provide students a means of instantly and effortlessly performing matrix computations, and thus free them to concentrate on what the computations mean, and when and why to perform them (Tucker, 1993). Many instructors use software in this context. Rather, students are intended to answer questions about what happens when certain computations are performed, without having to think too much about the mechanics of carrying out the operations. For example, students might experiment with the effect of scalar or diagonal matrices as multipliers, without actually performing all the matrix multiplications by hand. Most instructors feel that doing some of the matrix multiplications by hand provides insight about why results appear as they do. But, we can say that the ability to rapidly investigate a large number of examples makes a contribution to understanding.

Software can provide helpful visualization with two and three dimensional graphics. The ATLAST project provides a number of excellent tools. For example, the program span plots in 3 space the images of a large number of vectors, under multiplication by a particular fixed matrix. Software can be used to create interactive environments in which students can explore and experiment with vectors, matrices, transformations, etc., with graphical, symbolic, and numerical representations. Usually students must learn some syntax to use the software, but it is possible to free them from that by creating activities with a windows-style point and click interface. Matlab supports this kind of development, and to a more limited extent, Maple and Mathematica can be used in a similar way. There is a version of Mathwright that is available for free on the Internet, along with sample activities for students (Mathwright, 2008). The library includes a few linear algebra activities.

Connections with other mathematics courses

Guershon Harel (1997) has pointed out that calculus rests on a foundation of several years of background study at the secondary level, while linear algebra demands mastery of a number of critical ideas with little or no prior foundation. He goes on to propose that students be exposed to linear algebra at the secondary level, so that in college they have a suitable basis for abstraction and continued study.

What about students' prior knowledge is important for linear algebra teachers to consider? At the most concrete level, it is important to find out what linear algebra topics your students may already have seen. Have they worked with vectors, lines and planes in R^3? How many have already used matrix inversion on graphing calculators to solve linear systems? How many have worked with row operations? How do the answers impact your decisions about what to include in the course, and how long to spend on each topic (Cowen, 1997)?

The apparent connections between linear algebra, calculus, and differential equations are understandable by students. Also, it can be seen some contact points of linear algebra with other areas such as statistics and abstract algebra. The question of how to organize these connections is determined by linear algebra teachers. However, the prerequisites for linear algebra definitely affect what topics students can learn and how they can use them. If multivariable calculus is a prerequisite, then ideas about two and three dimensions that were presented in the calculus course can be more easily generalized in linear algebra. If students are required to see linear algebra first, then it can be used freely in calculus in the discussion of topics such as derivative and continuity (Katz, 1995). At many departments, neither of these courses is a prerequisite for the other. In that case, concepts like linearity of functions must be developed independently as needed in each course.

Conclusions

How to teach linear algebra is an important research question. We all know how to teach this course, and none of us do. We all know in the sense that we all have a good idea what we will do the next session we are scheduled to teach that course. But we need to understand better how students learn, and to aware that the appropriate content, strategies and context will be different in different settings. There is no one right way to teach that course, and there are some issues that may not be definitively resolved. We hope that this article has had a useful effect on the reader, and that the references may provide resources for further study.

References

Anton, H. (1973). *Elementary Linear Algebra*, New York: Wiley.
Bogomonly, M. (1999, December 21). Racing Students' Understanding: Linear Algebra. Retrieved from ftp://192.43.228.178/pub/EMIS/proceedings/PME31/2/65.pdf

Carlson, D. (1993). Teaching linear algebra: must the fog always roll in*? College Mathematics Journal*, *12*(1), 29-40.

Carlson, D., Charles R. J., David C. L., & Porter, A.D. (1993).The Linear Algebra Curriculum Study Group Recommendations for the First Course in Linear Algebra. *The College Mathematics Journal*, *24*(1), 41-46.

Cowen, C.C. (1997, July 21). *On the centrality of linear algebra in the Curriculum*.
Retrieved from www.maa.org/features/cowen.html

Day, J.M., & Kalman, D. (1999, July 15*). Teaching Linear Algebra: What are the Questions?*
Retrieved from http://pcmi.ias.edu/1998/1998-questions2.pdf

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages.
Journal of Mathematical Behavior, *5*, 281-303.

Day, J.M. (1997). Teaching Linear Algebra: New Ways. In Carlson D., Johnson, C, Lay, D., Porter, D., Watkins, A, & Watkins, W. (Eds*.), Resources for Teaching Linear Algebra*, (pp.107-126). Washington: MAA Society.

Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000).The obstacle of formalism in linear algebra. In Dorier, J.-L. (Ed.), *On the teaching of linear algebra*, (pp. 85-124). Dordrecht, the Netherlands: Kluwer Academic Publishers.

Dubinsky, E. (1997). Some thoughts on a first course in linear algebra on the college level. In Carlson D., Johnson, C, Lay, D., Porter, D., Watkins, A, & Watkins, W. (Eds*., Resources for Teaching Linear Algebra,* (pp.107-126). Washington: MAA Society.

Halmos, P.R (1942). *Finite Dimensional Vector Spaces.* Princeton: Van Nostrand.

Harel, G. (1997). The Linear Algebra Curriculum Study Group Recommendations: Moving Beyond Concept Definition. In Carlson D., Johnson, C, Lay, D., Porter, D., Watkins, A, & Watkins, W. (Eds., *Resources for Teaching Linear Algebra,* (pp.107-126). Washington: MAA Society.

Harel, G. (1998). Two Dual Assertions: The First on Learning and The Second on Teaching (or Vice Versa*). American Mathematical Monthly*, *105*(6), 497-507.

Harel, G., & Sowder, G. (2003, June 22). Students' Proof Schemes: Results from Exploratory Studies. Retrieved from http://class.pedf.cuni.cz/katedra/yerme/clanky_expert/Harel/Proof.pdf

Herrero, M.P. (2000). Strategies and computer projects for teaching linear algebra. *International Journal of Mathematics Education and Science Technology, 31*(2), 181-186.

Howard, A. (1997). *Elementary Linear Algebra with Applications*. New York: John-Wiley & Sons.

Lay, D.C. (1994). *Linear Algebra and its Applications.* Reading: Addison-Wesley.

Katz, V.J. (1995). Historical Ideas in Teaching Linear Algebra. In F. Swetz, J. Fauvel, O. Bekken, B. Johansson, & Victor Katz, (Eds.),*Learn from the Masters.* Washington: MAA Society.

Kolman, B., & Hill, D. (1999).*Elementary Linear Algebra with Applications.* New York: Kluwer Academic Publishers.

Lawrence, E.S., Insel, A.J., & Friedberg, S.H. (2008*). Elementary Linear Algebra*. USA: Pearson Education.

Mathwright, (2008). *Effective mathematical software*. Retrieved from: http://www.mathwright.com

Schneider, H, & Barker, G.P. (1968). Matrices and Linear Algebra, New York: Dover Publications.

Strang, G. (2005). *Introduction to Linear Algebra*. USA: Wellesley-Cambridge Press.

Poole, D. (2007). *Linear Algebra: A Modern Introduction.* London: Brooks Coole.

Richard, P.C. (1997). *Linear Algebra: Ideas and Applications:* Amsterdam: Willey-Interscience.

Tucker, A. (1993). The Growing Importance of Linear Algebra in Undergraduate Mathematics. *The College Mathematics Journal*, 1, 3-9.

Waerden, B.L. (1936). *Algebra*. Berlin: Springer Verlag.